FINITISM, DISCRETENESS, RECURSION, AND RANDOMNESS

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ABSTRACT: This paper reviews the results of work spanning more than a decade of research into empirical modeling methodologies. During that time, deficiencies and inconsistencies have been encountered in standard methods which forced the development of a fundamentally different approach. Some of the problems encountered, the methodology developed by the author, and some of the applications of the methodology are reviewed.

Learning to Add

One-half plus one-half is equal to one. Or is it? If we add a half an orange to half an apple, what is obtained? We might answer, "some fruit"; but notice the jump that has taken place in thinking during this operation. Initially, there are two classes of objects (half oranges and half apples). Then we are forced to jump to the creation of a third class of objects. We can make the jump stronger, for if one adds a half an orange to a half a room, what is obtained? I am certain that we could name this "object", but the notion is unfamiliar and somehow different from what we usually think of when we go through the operation of addition. We feel much more comfortable with adding two orange halves. And yet if the two orange halves are not alike in all aspects, what is obtained? What then do we mean by "one whole orange"?

Ignoring vitalistic arguments, this is precisely the concern that made King Solomon's wisdom work so well when he suggested that the two claimants to the motherhood of a child each accept half of the child. The fact is, that our world is composed of objects which have multiple identifying qualities. If these qualities are not evenly distributed through an object, then partitioning the object becomes rather difficult. Partitioning changes the nature of the object such that the parts must be renamed (i.e. in general, partitions belong to a different class than does the whole object).

The notion of doing arithmetic with decimal fractions is deceptive and seductive. It is essentially incorrect in a discrete world. The assumption is made that all qualities are equally distributed (homogenous), are therefore equally divisible, and furthermore have no fundamentally smallest unit. Partitioning of discrete objects (sets, spaces, etc.) results in objects of a different class than the original objects. Each division operation, such as those which occurs in converting a rational to a decimal fraction, creates another class of objects in general. Thus a decimal fraction represents not the cardinality of one class of objects, but many, all hierarchically organized. And irrational fractions carry this process to extremes, namely, to infinite classes of objects. But continuum mathematics makes no distinctions, and no provision for preserving distinctions of class.

This lack of rigor carries over into interpretations of statistics and randomness. Probabilities are generally expressed as decimal fractions having values between 0 and 1, the process of normalization. However, it is not the same to refer to one out of every five as to two out of every ten. Knowing two out of every ten tells us nothing about the partitioned set's distribution function, so we are not assured of the one out of every five. And, of course, the definition of randomness involves infinity - either in the continuum or a denumerable infinity - which is itself not definable constructively.

Though counting, measurement, and arithmetic often appear in close association in mathematics, physics, philosophy, etc., etc., these have no clearly defined mathematical association without additional assumptions which are usually unacceptable in the given context. For example, there is no definition of mapping or function without the notion of an open neighborhood and discrete spaces can have no open neighborhood. Without a definition of function, there is no distance function and no metric. If one chooses the discrete topology, no set ever has a near point in the closure of the set, thus every set is both open and closed! On the other hand, no set is connected (except one point sets) and all finite sets are compact. We do not mean to imply that a mathematics could not be constructed which would eliminate these problems, only that the accepted foundations of mathematics does not serve the need.

The mathematics used in economics, statistics, physics, computer science, and engineering is filled with hidden infinities, notions of continuity, limits, open neighborhoods, and even randomness. We must not assume continuity while performing digital operations. No matter how sophisticated the sophistry, paradoxes are bound to result due to the unacknowledged infinities. Understanding this situation is particulary important in logic, empirical modeling, psychology, and quantum physics. Work in linguistic logic (*) and natural language acquisition (*) has demonstrated the existence of discrete and non-commutative structures, yet the very idea of language production or learning is a process. Empirical models of such processes frequently demand a means of expressing notions of discrete dynamic generation based upon static topology: each incremental generation must alter the topology and require yet another generation based on some incomplete quality (closure, representational completeness, etc.). Thus the representation must always be incomplete according to the selection algorithm and always complete by the structure algorithm. But it is all too easy to assume that all is well in the foundations, only to labor intensively trying to resolve the paradoxes that must result in some abtruse form.

We are placed in an awkward position: how can we consistently define counting, addition, and division? How can we talk with integrity about the integers and the real number line, the discrete and the continuous, the finite and the random? Indeed, how do we go about making models of the world without encountering devastating paradoxes and how do these paradoxes occur?

Problems with Models

Analogy and metaphor are the most powerful techniques available among the aids to communication and education. Yet, these are but the imprecise, often misunderstood and misused, relatives of the model. Nothing restrains us from using models in place of analogy and metaphor more than the fact that the operational definition of a model and its relationship to theory has not been worked out in technical and mathematical detail. A previous paper (*) provided a foundation for that technical detail and gave some hint of the power of the tools. Specifically, we examined general notions of modeling, the relationship to theory, a scheme for classifying models, and some of the reasons that models fail. Our procedure described models as recursively generated structures and drew heavily on concepts from differential topology. We were able to show how hierarchical models arise and how such models are susceptible to paradox.

In early attempts to secure a consistent modeling methodology, several difficulties were apparent. Suppose that one defines a model as a collection of formal symbols and associated rules of manipulation together with a set of rules of correspondence mapping the formal symbols to elements of the set to be modelled (usually called the observation set although object language is perhaps a better terminology). Clearly one wishes to be able to describe an observation set in several different ways. There are relationships between these various models which provide both local and global structure. This suggests the existence of transformations between the various views or descriptions (models) of the observation set.

One demands of a good model that the relationship between elements in the observation set be mirrored in the relations between elements of the formalism. This provides us with a description of the local structure (or topology); it does not provide the usually more interesting global properties. For example, while a good model of flora could provide us with a highly ordered and useful catalog, it will not, in general, give a hierarchical classification nor will it provide connections to the rest of the world. Ordinary models provide descriptions based upon intrinsic orderings but fail to provide either higher-ordered relationships within the class or, what is more important, relationships between models of different classes (i.e., global topology).

What is needed then, is a methodology which gives the intrinsic orderings, the hierarchical orderings, and the connections to other orderings and other classes (models of other observations sets) in a consistent, single formalism. Furthermore, the formalism must not impose specific structure. That structure will be imposed is not the question. Rather, we suggest that the test of an appropriate structure is a test of the validity of the model. Given a sufficiently powerful formalism, one may choose the structure that one imposes and, the range (local or global) of that structure as well.

One might suppose that the usual logics (predicate logic or the lambda calculus) would provide such a formalism. The difficulty with formalisms of this sort lies in the general inability to move from orderings defined on inclusion relations (strictly speaking a quasi-ordering) to orderings which admit a distance measure or metric. This leads to an impasse as regards models expressed in formalisms such as those given in linear differential equations. If one chooses a formalism which gives a distance measure 'a priori', then one is, of course, met with an impasse from the opposite direction - it becomes difficult to think of models (a la Tarski) of objects on which one does not usually impose a distance measure.

In essence, we seek a formalism which is at once powerful enough to describe highly structured and unstructured objects, with and without distance measures, both finite and (constructively, at least) infinite. Furthermore, the methodology itself must reflect the process aspects of the system being modeled. We do not accept static, complete models as meaningful. Certainly this will be some form of a topology and specificly one which will provide a geometry when a distance measure is chosen.

The differential topology will not do, as the sets involved are already chosen to be locally infinite and continuous. The point set topology has the opposite difficulty - it presents no consistent means of getting to the usual continuous structures. Combinatorial topologies are not applicable to models which must have a metric. Overall, the existing topologies provide no clear means of specifying the orderings which provide structure in a model.

They are founded upon set theory, which already embeds far too much structure... sets are characterized by both cardinality and ordinality and the ordinality is given only by a number, there being no information provided as to what structure was assumed (or imposed) to obtain this ordinality. In view of this dilemma we would construct a topology with the required properties.

A Finite Approach to the Differential Topology

Early forays into the breach that is paradox arose from an attempt to treat formalisms (and therefore models and theories) more generally than presented in the "classical view". The explanatory modeling methodology (*McGoveran, 1979) proposed treating interdisciplinary models from a triune perspective: an epistemology (rather like a pragmatics), a semantics, and a procedural syntax. The semantics of this methodology imposes a certain set theoretic structure on the model. At the same time, the procedural syntax of the methodology imposes a morphological structure by allowing for maps between the pragmatics, semantics, and syntax. These characteristics of the methodology - i.e., sets with mapping relations - suggest that a kind of differential topology might be operative in the modelling process. Unfortunately, the sets involved are rarely connected in the usual sense, nor are they continuous sets (sets of infinite cardinality). We found that the methodology made paradoxes painfully obvious.

When one speaks of creating a model of a physical system, one begins the formal model with a definition of the observation set. The observation set, of necessity, is finite. This is true even if one assumes that the underlying space is continuous and that the population being sampled is infinite, although we reject such a contention as unfounded and unfoundable. Nonetheless, one generally assumes that the quantitative formalism should be inherently capable of expressing a (apparently) continuous range of values (exceptions include certain aspects of quantum mechanics). The observations are associated with a set of representations. This representation set constitutes the formal language of the model. When one examines this situation in terms of maps from an observation set to a representation set, inconsistencies soon arise from the classical definitions of maps, sets, spaces, manifolds, etc. A map can be defined only over some local topology. This toplogy is given by the way in which one forms an atlas from a set of charts consisting of an open neighborhood and its map. But a set of observations (or representations) is discrete: it can have no open neighborhoods. Thus the mapping fails if treated in any way other than as a simple association between discrete sets. There can be no coordinate systems, no distance functions, no metrics, no generalized transformations, no manifolds, etc. Nevertheless, most models do represent the observation set as having such structure. Indeed, applied mathematics is useless without such assumptions in contradiction to the interdicts of pure mathematics. The major difficulty lies in the contradictory assumptions of the continuum in the concepts to be used and the discreteness of the space to which the concepts are applied: no consistent treatment has been available.

Recursion

In order to preserve the procedural aspect of the syntax, one demands that any solution to these difficulties provide a foundation compatible with the theory of recursive functions. This allows us to maintain a procedural or process view. For purposes of analysis, the process must be terminated. For purposes of generating the model consistently, the termination must be arbitrary. To treat the observation set or the set of formal symbols and operations or the set of correspondence rules as static entities is to miss the point of a modeling methodology. Casting a modeling methodology in recursion theoretic terms helps us to remember the dynamic nature of empirical, formal modeling.

Moreover, the resulting modeling methodology suggests, less than formally, that the occurrence of paradoxes can be relegated to the domain of hierarchical structures which contain an incompatibility. The "fix" suggested is not one of eliminating so-called self-reference a la a Russellian theory of types, but rather to evolve a formalism for dealing with hierarchical structures (more closely akin to Frege's levels) which is sufficiently precise to account for the paradox and uncover the hidden contradiction (or incompatibility). Part of the nature of paradox is the tendency toward forms of infinite regress; structures for which, given a truth value of a conclusion, one could not obtain consistent tentative truth values for the premises. Such structures lead to viable criticisms of the notion of the completed infinite: these are essentially the constructionist arguments. Thus it is also important to provide a formalism which does not depend in any way on infinities - we demand a constructive, finite formalism which is rich enough to provide the usual large cardinalities observed in nature through recursive enumeration.

The formalism thus outlined demands a finite approach to concepts introduced in the continuum. In short, one desires a formal procedure whereby local topology (structure) can be introduced on finite collections of arbitrary elements. Furthermore, one desires a sort of mathematicians correspondence principal: namely, in the limit of collections with large cardinality one should recover the standard definitions given by the continuum approach.

Differential topology, if it were inherently capable of dealing with finite sets, and if it were couched early on in terms which made the various ordering relations explicit, would be the kind of formalism required for a coherent modeling methodology. The power of speaking of the observation set as a manifold is obvious. However, we may not do this when the sets are finite (all our physically interpretable observation sets are finite) with a clear conscience: we have no finite set definition for open neighborhood, no finite set definition for continuous, not finite set definition of the role played by Rn in the definition of a manifold.

The differential topology was redeveloped with the following constraints in mind:

1. All objects are to be given without appeal to the completed infinite or to the completed infinitesimal.

2. The mathematics thus defined are to have the usual properties for collections of objects with sufficiently large cardinality.

3. The formalism is to be free of interpretation - i.e., it is only a typography... a (hopefully) consistent way of manipulating symbols.

4. The entirety is to be recursively constructed. (This will ensure computability when a distance measure is provided.)

The details of this approach and definitions of the terminology used in the rest of this paper can be found in (*).

A Discrete Constructive Modeling Methodology

We halt the infinite regression of analysis of terms in modeling by recognizing the Epistemological framework. Namely, we always bootstrap into the modeling process with a set of loose agreements and definitions. We don't really know what we are talking about. But recursion theory gives us a consistent mechanics of typography and the procedural framework gives us a recursive method of getting the "right" model and definitions. In some sense we thus have a "fixed past and uncertain future".

Having thus given the foundation for a finite differential geometry, we proceeded to a foundation of modeling methodology.

Df: A MODELING METHODOLOGY consists of three broadly defined structures: An EPISTEMOLOGICAL FRAMEWORK, A REPRESENTATIONAL FRAMEWORK, and a PROCEDURAL FRAMEWORK.

Df: An EPISTEMOLOGICAL FRAMEWORK is a set of loosely defined agreements made explicit by those injecting information into the model formulation.

1. agreement of cooperative communications
   * commonly defined terms as fundamental
   * fundamental vs. derived terms
   * agreement of pertinence
2. agreement of intent
3. agreement on observations
4. agreement of explicit assumptions
5. The Razor
   * agreement of minimal generality
   * agreement of elegance

Df: A REPRESENTATIONAL FRAMEWORK is an abstract formalism consisting of a set of symbols and a set of rules of manipulation.

Df: A PROCEDURAL FRAMEWORK is an algorithm which serves to establish rules of correspondence between the observations (as agreed upon in the E-frame) and the symbols of the R-frame, and which then, through recursion, serves to modify the rules of correspondence and the E-frame and R-frame until a sufficient level of agreement concerning accuracy is achieved or the model fails (a la Kuhn).

Thus we see a relationship between two sets being established (the O and F) with two sets of rules for modification and/or information extraction.

We now cast this in terms of the finite differential geometry.

Df: An OB SET is a collection of observations. The obs are differentiated (altered from SORT to SET) by one or more ordering relations which serve to establish the lattice structure of the obs.

Df: An OB SUBSET is a set of obs, each of which are ordered by the same relation. (They may be mulitply connected) multi-ordered

Df: A SORT of FORMAL SYMBOLS is a collection of labels which may be ordered (converted to a SET) by a set of rules of manipulation. The SET of FS may be closed under the rules or open i.e., finite or infinite (increasing monotonic recursion). Generally, this serves to form a combinatorial system.

Df: A RULE OF CORRESPONDENCE is a binary map between an element of F and an element of O.

Df: A PROCEDURE is a recursion algorithm which (a) provides a recursive and exhaustive enumeration of the elements of O and the elements of F such that there exists a continuous map between O and F in the sense given above and which (b) provides a recursive re-parameterization of the map such that there exists a 1-1 map between a subset of O and a subset of F.

Ideally the cardinality of these subsets increments with each recursion up to the cardinality of O itself.

Observation Space

We begin with a number of observations which may be clustered (grouped into prearranged classes) into sets O_i; these observation sets are said to cover the observation space O in the sense that $\cup_i O_i = O$. Because our O must have boundaries (for any hypothetical O_u; O_u is a subset of O) and is discrete, O is non-Hausdorf. (Since in the usual topology all metric spaces are Hausdorf, our space is not metric is the usual sense.)

Clearly, for any finite O, there are finite number of possible disjoint partitions of O, namely n! where n is the cardinality of O. However, the partitions need not be disjoint - we allow dependent observations and any ob to be in more than one partition. Thus the number of partitions is potentially infinite.

We map each partition O_i of O to some subspace S of R^N by some map R^N. If each such subspace S^M of R^N is arbitrarily "labeled" with some formal symbol f_i, then the partitions O_i of O may be taken as "objects" in O_i and referred to by the f_i. The R^N then form rules of correspondence.

We define relationships between the O_i objects in terms of the coordinate transformations between the S^M.

Models

An examination of our definitions tell us immediately that there is no parameterization on S which gives special coordinates. In fact, there is no structure at all on S without a parameterization. There exists no metric, only local topology induced by f; global topology is given by cardinality and partitioning as well as coordinate transformations between partitions giving connectivity.

I. Having thus defined the basic concepts of differential geometry we can proceed to use differential geometry on a discrete manifold to give the mathematics of modeling.

1. Choose the ob. set with n elements - a rec. enum. set with cardinality n.

2. a. Define the n obs (labels) preferably discrete - partition the set into disjoint subsets

   b. Choose a set of symbols for these partitions - label the partitions

3. Choose a set of Rules of Correspondence to a Formalism - map to some space such as Rn (can always choose Rn locally for map) must define objects on open sets though.

4. Determine relationship between obs vis-a-vis formalism - establish set of coordinate transformations and determine invariances for global properties!

5. This establishes model if set of obs cardianlity is fixed. Otherwise recursion is required.	- recursion alters partitions, range of maps, coordinate parameterizations, maps, etc.

Hierarchical Models

Hierarchical models may be defined as follows:

6. Start with a model	
7. Do a many-to-one map from formalism to ob labels.	- redefine the partitioning by refinement, map from image to representation set with new partitions using inverse map. This insures consistency for next step.
8. Remap to formalism.	- map from new partitions to image space using old map.
9. Keep in mind constraints of many-to-one map - this map provides inclusion relations on the Representation set, thus partitions contain partitions or parts thereof forming a lattice of partitions.	

Paradox

At step 7 above, leave some sub-partition uncovered by the new partitioning or redefine partitions so that it is locally (somewhere on S) only a permutation of the partitions i.e., map is not locally many-to-one and map is not the inverse map for that used in defining a model (3 above). Use this fact to give an algorithm for paradox. This gives all we need for paradox except some secondary formalism - a second recursive enumeration or labeling.

In group theoretic terms one would say that the homomorphism fails - is not defined for all points and is not many-to-one. This is because the hierarchical re-partitioning must establish a covering (previous partition covers lower level partition) and a covering must be a homomorphism of the two groups involved.

To put the procedure for generating paradox in simple terms:

1. Identify the system in which you would like to have the paradox.

2. Now identify a system which will allow you to describe the first system. Note that the two systems may be the "same" to some degree of approximation.

3. Be more specific. Make certain that the "second" system and the "first" are not quite compatible (i.e. that one contains a contradiction of at least one element of the other).

4. Re-interpret the "second" system in such a way that only a portion of it is required to talk about the "first", noting the way in which this establishes self-reference for the "first" system.

5. Insure that the contradiction noted in 3. lies within the portion of the "second" system described in 4.

6. Define a statement about some arbitrary portion of the "second" system in terms of the "first" system, but referring to the incompatibility in a primary manner (as a precedent to a consequent or as a direct object perhaps).

7. This statement will be a paradox.

There have been many attempts over the past eighty years to treat paradox in a coherent fashion. Since the communication to Frege by Russell of the contradiction in naive set theory (Russell's paradox), attempts to formalize representatons and resolutions of the logical paradoxes have played a central role in mathematics and philosophy. The various explanations of the paradoxes which plague formalisms and the several more widely known general proposals to remove paradox in such a way as to retain the more useful parts of formalism, have each failed in one way or another. The first attempt to eliminate a paradox from a formal system was Frege's weakening of his comprehension axiom. While this blocked the straightforward derivation of Russell's paradox, Lesniewski proved that even the modified axiom yielded a contradiction. Russell's attempts to establish various interdictions (self-reference naively denied, the theory of types, and its variations) which would prevent the formulation of paradox while retaining the required foundations of mathematics failed. Skolem was of the opinion that paradox (and unnecessary complexity) could be eliminated by rejecting quantification over infinite domains. Nevertheless, his discovery (Skolem's paradox) provides the foundation for the Lowenheim-Skolem Theorem which philosophically denies the resolution. The proof of Godel's Second Incompleteness Theorem is perhaps the penultimate example of the importance of paradox. Godel, rather than attempting to resolve paradox, showed how to derive and represent a paradox within a formal system if it was sufficiently powerful to contain Peano arithmetic. The result was the end of the search for a consistent and complete axiomatic foundation for mathematics.

We have presented an explanation of paradox - and a general resolution - with a completely different approach. The results, however, are strikingly familiar, as they are related to the theory of types. Our explanation derives from a general treatment of the concept of model making - a theoretic modeling methodology. Such a general treatment is thwarted if one insists on the usual definitions of concepts such as set, open neighborhood, map, number, etc. We are forced to deal with discrete spaces, and at the same time, insure that the usual definitions of concepts in differential topology are obtained when one passes over to 'infinite' sets (continuous spaces). This is essentially a strengthened version of the approach of Skolem. Whereas Skolem rejected quantification over infinite domains, we reject the infinite domain as well. It is our position that 'an infinite domain' can have no Bedeutung or reference and is thus different from the proper name that is a number. Thus, one can not reason about 'infinity' or 'orders of infinity' using the mathematical and logical apparatus designed for dealing with numbers. The standard notion of limit fails to distinguish between these concepts and is rejected as a consequence. Even the otherwise recursive generation of the natural numbers by von Neumann makes use of a sequence of Dedekind cuts of the (infinite) real number line. However, the logical calculus involved may be taken as devoid of such concepts if the generation is given over a discrete set. Then the generation is 'up to' the finite cardinality of the set.

Following Gian-Carlo Rota, we "... provide a systematic technique for setting up (other) algebras of generating functions suited to particular enumerations. Our initial observation is that most familiar discrete structures, while often devoid of any algebraic composition laws, are nevertheless often endowed with a natural order structure. The solution of the problem of their enumeration thus turns out to depend more often than not upon associating suitable computational devices to such order structures..." Unlike Rota however, we are not satisfied with a language that is dependent on an embedding of discrete structures in a continuous one. We have developed terminology afresh, without the taint of a continuum (and infinities) in which we can have no faith. Thus, while this work has some relationship to the reduced incidence algebra, we believe that it is sufficiently different to avoid the paradoxes of the infinite.

APPLICATIONS

Godel's Theorem

Since the publication of Godel's famous paper in 1931 entitled "On Formally Undecidable Propositions of Principia Mathematica and Related Systems I", the proposal that classical mathematics be formulated as a formal axiomatic theory, and that the theory should be proved to be free of contradiction (Hilbert, 1904), has not been the subject of any serious and concerted effort within the mathematical and philosophical communities. Analysis demonstrates that paradoxes are based upon (sometimes obscure) contradiction of reasoning or interpretation. This paper will re-examine the proof of Godel's theorem inasmuch as it is the major cause of the demise of Hilbert's Program and is an example of an important result based upon paradox. Certain assumptions in the proof of Godel's Incompleteness Theorem will be shown to be contradictory and that this contradiction defeats the validity of the proof.

Formalism or formal axiomatics suffered a severe blow with the publication of "On Formally Undecidable Propositions of Principia Mathematica and Related Systems I" (Godel, 1931). The achievements in this landmark paper were numerous, from the development of a method for encoding the system of Principia Mathematica in arithmetic (elementary number theory) to demonstrations that forty-five number-theoretic predicates are primitive recursive. The key theorem of the paper (Theorem VI) was to become famous and the construction of the proof, unique and unusual though it was, to be deemed flawless.

Stated simply, the theorem shows that any system capable of representing elementary number theory would necessarily include undecidable propositions and thus, demonstrated that classical mathematics could not be proved to be consistent--i.e., free from contradiction. This result ended the search for a proof that classical mathematics could be completely axiomatized in a consistent formal system (Hilbert's Program, 1904).

The argument used by Godel is closely related to Richard's antinomy and to the Liar. However, it is our thesis that all paradox is due to a contradiction. The sources of the contradiction are categorizable into three types: (a) the interpretation contradicts a rule of the logic system, (b) the argument involves a hidden contradiction between some seemingly arbitrary and harmless assumptions, or (c) the argument conclusion involves an interpretation based on completed infinities (or infinitesimals), the concept of which we take here to be a contradiction in and of itself. Most of the more famous (and more difficult to resolve) paradoxes involve another factor. Frequently it is called self-reference. However, self-reference is a specific instance of a more general type of structure, namely hierarchical structures. A hierarchical system may be characterized mathematically by a lattice of partitions (partially) ordered by refinement. (NOTE: all systems exhibiting self reference are not strictly hierarchical in that the blocks of the partitions may not be disjoint, but are in fact of non-empty intersection. Nonetheless, it is possible to establish a hierarchical system whenever such a system exists.) Self-reference occurs when, for a suitable map, one or more blocks of a partition in the lattice is mapped to a different partition in the lattice. Then, the block or blocks in question have multiple interpretations, as both the domain and the range of the map. Of course, such a map lies outside the lattice structure. If multiple hierarchical systems are involved as in indirect self-reference, then multiple maps are required to establish the self-reference, at least one such map for each system (for details, see (*). There is, of course, nothing to prevent the map in question from being a familiar operation within the system. This simply confuses the nature of the map. The fact is that the map, in establishing self-reference, violates the antisymmetric postulate and thus destroys the partial ordering.

Paradoxes which contain contradictions of type (a) are the so-called common fallacies. Those of which contain contradictions of type (b) are frequently more difficult to resolve as the contradiction generally has to do with restrictions on the domain of the argument (or universe of discourse). Paradoxes which involve contradictions of type (c) can also be subtle, in that it is not always easy to see how the completed infinite is assumed. Arguments which use Cantor's diagonal method run the risk of demanding that a system be both finite and infinite when the critical proposition is interpreted (Cantor did not make this error). Zeno's Paradox would have us accept the continuum embedded within the discrete or vice-versa.

Godel's Theorem contains a hidden contradiction of type (b) above. It is given here as a paradigm of what can go wrong when a system admits of multiple interpretations and where self-reference if abundant. Even so, there would be no difficulty were it not for the (hidden) contradiction.

Demonstrating the truth of this thesis is difficult. In what follows, do not be misled into believing that we will discover a line which directly contradicts another line of the proof. The encoding is much too complex. Instead, we will point out the two lines of the proof which are not directly derived by some rule of the system from a previous line. Indeed, the lines involve the choice of an instance of a universal. Under ordinary circumstances, this choice would be free: if the universal is true then certainly the instance is true. However, two points are important. First, the system and the proof are purely constructive. Thus the domain of the universal is some finite list of objects. Second, either of the two choices would be free if they were not coupled in any way: if the instance chosen for the one did not affect the domain of the universal in the other. It is our task to show that the universals and the instantiations are coupled. If the instantiations are truly uncoupled, then the truth values of the instantiations can not affect the outcome of the proof. For example, consider the following:

All men are mortal.
1 Socrates is a man.
Socrates is mortal.

All ancient philosophers are liars.
2 Socrates is an ancient philosopher.
Socrates is a liar.

All men are liars.
3 Socrates is a man.
Socrates is a liar.

4 Socrates is mortal or Socrates is a liar.

In the system consisting of arguments 1 and 3, clearly, if it is denied that Socrates is a man, 4 is false, but true otherwise. In the system consisting of 2 and 3, if it is denied that Socrates is an ancient philosopher, the truth of 4 is unaffected. This demonstrates what we mean by coupling. The domains of the universals of systems 1 and 2 are coupled. The domains of the systems 2 and 3 are not coupled. Of course, the coupling in Godel's proof is much more subtle since the proof is given at two levels: number theory and <u>Principia Mathematica</u>. Nonetheless, once we have identified the candidate propositions, we can determine whether or not they are coupled by negating the instantiation of one of them and determining if the conclusion of the proof is affected. If it is, then we must determine if this is due to a coupling with a direct line of the proof or with another instantiation. In particular, between the instantiation of any two universal quantifiers, their are four possible cases resulting from the combinations of affirming and denying the instantiation throughout the proof. If the predicates are coupled, it must be the case that denying one of the instantiations leads to the contradiction of the original conclusions while affirming both recovers the original conclusion. If any other result obtains, then either (a) one or other of the predicates is coupled to a direct line of the proof (but not both), or (b) the affirmation or denial of the instantiation has no affect on the proof and is thus a "dummy" variable.

<u>The Diagonal Method</u>
The diagonal method as introduced by Cantor in his proof of the existence of an uncountable set and that, in fact, the irrationals were such a set, (Cantor's second diagonal method), has been extended by recursion theorists to questions of computability or decidability. While the interpretation given by Cantor is not at issue, we must object to the method as used in computability theory.

By a suitable choice of a system from those systems which should be susceptible to the diagonal method, it is possible to show that the diagonal method is a flawed method of argument. The method has been applied in proofs of Godel's Incompleteness Theorem among others and yet, by demonstrating that it gives wrong results in one case, we show that it is not trustworthy in the mode commonly used for decision and halting problems by recursion theorists.

We take as a starting point the assumption that all effectively computable procedures are representable in terms of a Turing machine. Consider all strings of a given system S that have been thus far constructed. Assume that there is a Turing machine that serves to generate these strings. (Each instruction of a Turing machine can be given by a quadruple of the form internal input state n, input operation a, output operation b, internal output state m.) For each string, there will be a set of instructions (a function) which serves to cause the Turing machine to generate the string. We can order these sets of instructions by ordering the output strings. Namely, interpret the string as a binary number. Order the list of output strings by this index in increasing order. Now, create a catalog index (Godel numbering) which simply enumerates the strings - i.e. label the first string "1", the second string "2", etc.. Let each such label represent the set of instructions necessary to generate the given string. All that we have done is effectively computable. We now have an enumerable list of functions which generate the system up to output string N.

Now assume that there exists a decision procedure for the system S. It follows that there exists a function f which takes as input the index for an entry in the set of instructions and which generates, as output, the string which would be produced by the set of instructions. The decision procedure is but one step beyond this - namely, from the function f, we can construct a decision procedure h. The function h takes as input a given string and outputs a 1 if the string is one that will be generated from the current catalog of functions and outputs a 0 otherwise. That this is possible given the function f is clear: h uses f recursively to generate the strings represented by the catalog, comparing each string to the input string until it finds a match or until it terminates - i.e. halts.

Let Q_x be the (x+1)st derivation or set of instructions.

Let f_x (I)

be the function corresponding to Q_x. However the function f does not exist. Assume there is such a recursive f. It can be used to define a new partial function g as follows:

$g(x) = f_x(x) + 1$ (II)

Evidently, we have an algorithm for computing g; namely, to get h(x) for a given x, generate the list of derivations out of Q_x then employ Q_x to compute $f_x(x)$, then add one. On the other hand g cannot be partial recursive. If it were, we would have h = g_x for some x. But then, we would have

$g_x(x) = h(x) = g_x(x) + 1,$ (III)

a contradiction. Since g is not partial recursive, and the operation of adding one to a partial recursive function is a partial recursive function, it follows that f is not partial recursive - not effectively computable as was assumed.

Since it would be difficult to argue that the constructiveness (effective computability) of the system S is not itself a valid assumption (as assumed), it would seem to follow that the decision procedure does not exist. However, notice that we have not provided an instance of a system S nor an interpretation of the system S. We now do so. Let the system S consist of the symbols s and ' and the following rule of inference:

for all x, x = S ---> x' = S

This system shall have two isomorphic interpretations:

1) function interpretation:
   s is the zero symbol
   ' is the operation of adding one (successor function)

2) sentential interpretation:
   s is the initial string of S ("This is a string of S")
   ' is the operation of enclosing in quotes and concatenating "is a string of S".

Following Godel, both of these interpretations are partial recursive. It follows that formulas (I) and (II) are, by interpretation of S, functions of S since they involve only the operations of adding one to a given function of S. Therefore, the contradiction (III) must lie in other than the assumption that f is partial recursive and thus the diagonal method fails.

The reason for the failure is now clear. We generated a system S for our enumerable list up to string N. The diagonal method, however, argues by induction beyond N (formula (II) above) to a completed list where N is infinite. Hence, the diagonal method is neither constructuve nor a valid method of proof in the context of decision problems.

It is interesting to note that the system constructed above is exactly the type of system demanded for empirical modeling of self-organizing systems. At any point, it is ALMOST provably decidable within the system and obviously so outside the system, only the proof conclusion line is not part of the system (yet). It is recursive and consists of a generative aspect (enumeration) and a structural aspect (decidability).

Random Number Generators

It has long been accepted that computability is synonymous with the capabilities of a Turing machine (Church's Thesis). Shannon (1967) has shown that there is nothing that an infinite Turing machine can do, given a true random number generator supplying its input, that an infinite Turing machine with non-random input could not. The method of proof used by Shannon depends on the

Turing machine being infinite and, in fact, Shannon goes to great lengths to point out that the proof makes no statements about finite machines. However, we have argued that a finite Turing machine can not decide whether or not its input source is a true random number generator or a pseudo-random number generator.

Consider a system composed of three elements: (1) a universal Turing machine, (2) a finite memory, and (3) a number generator. It will be shown that such a system is incapable of deciding whether or not the number generator produces repeating binary strings of length n whenever the memory is smaller than an amount m equal to n + log n.

For suppose that the Turing machine takes as input a particular string of length n and we wish it to determine whether or not the number generator is producing this string repeatedly. The Turing machine must consume an amount of memory equal to n in order to store the string. It can then scan the output of the number generator, comparing it to the first symbol of the target string. First, we need a counter to point successive symbols of the target string. This will require an amount of memory equal to x such that n = (2exp x). Whenever the Turing machine detects the symbol pointed to by the counter, it increments the counter and continues scanning. If it detects a symbol not equivalent to the one being pointed to, the counter is reset to point to the first symbol. If the counter reaches the end of the target string (is set to all 1's), then the full string has been detected. The counter is reset to point to the first symbol of the target string and the scanning continues.

It follows that the system can not decide whether or not the target string has been produced if it has memory less than n + log n. But this means that the system can not distinguish between number generators which produce repeating strings and random numbers. Clearly, the symbols in the repeating strings will occur with equal probability, as required for a random distribution. However, since the system can not detect that a given string is repeating, it can not detect that some string of length n is repeating. Thus, for system with less than n + log n memory, a generator producing repeating strings of length n is indistinguishable from a generator producting random numbers.

CONCLUSIONS

By neglecting discrete sets in R^N, we are in no difficulty so long as we remember that defining the members of the image set must be recursive. Our recursion serves to maintain the class of elements in the set (insuring an equiv. class). If we use the continuous properties and make a Dedekind cut, we must simply remind ourselves that in so doing we have changed class membership (e.g., 1/2 is not in the same class as 0 and 1). Division or fractions imposes a lattice of partitions, and this indicates the presence of a hierarchical model. In a similar fashion, probabilities must be expressed with care: randomness is not a well-defined concept over discrete, finite spaces. And, in contradistinction to the usual view, we interpret analytic functions as approximations to the finite, discrete combinatoric functions required by our topology, rather than the other way around. Given the correct discrete unit and topology, the combinatoric summation provide more accurate information than does the analytic integral.

In a sense, the set of points S in R^N is the union f the image sets for all classes in the domain. Thus we may define a distance function on S. We may also define a distance function on a discrete set (a non-Hausdorff space) but it will be "multi-valued" in the sense that the ordering between elements need not produce a single chain: thus there may be more than one "path" between elements. If we take the distance functin such that the number of elements traversed is minimal, then at best we must assume that elements in the string defining the distance [(minimal simple chain) between any two elements] may in fact be twins under the equivalence class defined by the distance function. Clearly we do not care if the MAPS are strictly recursive; they may be analytic so long as we keep in mind the constraints on the space.

The formalism reviewed here has important results with regard to the combinatorial hierarchy. Of special interest is the susceptibility of the combinatorial hierarchy to paradox. It is extremely important that such models be consistently interpreted if one is to avoid paradox. After all, the other ingredient patiently awaits the unwary - the very nature of a hierarchical model is self-reference. In addition, this work has other results of importance for the combinatorial hierarchy. We have demonstrated ways to avoid decidability issues, concerns over algorithmic randomness, the problems encountered with distance functions on discrete or combinatorial topologies, and finally, interpretations of probabilities.

Paradoxes are resolvable. They are neither intrinsic to a system or model, nor can they be devastating to the joint consistency and completeness of the formalism. In another paper we have applied these results to locate and expose a syntactic flaw in the original proof of Godel's Second Incompleteness Theorem. Similar results may be expected to follow where the logician has assumed the legitimate embedding of a paradox in his proof schema. Hierarchical models are necessary for the representation of paradox. The levels of a hierarchical model may be interpreted as types or even more consistent with the present work, as levels a la Frege. It is the view of the author that hierarchical models are desirable and necessary. It will not do to exclude hierarchies nor to limit the types that may be used. Nor is it sufficient to eliminate quantification over infinite domains. Constructive formalisms which adhere strictly to finitism are necessary. This is due to the fact that one must not confuse types when the operations involved are type specific. Type specific operations abound in hierarchical systems: it is generally not possible to derive a consistent interpretation of an operation defined at a high level of a hierarchy when it is to be applied to a lower level. The reason is clear: the refinement which makes up a hierarchy is a partially-ordered partition lattice. The maps defined across refinement are many-to-one. There is no unique inverse. Thus, the concept of infinity may not be interpreted at the level of number and it is fallacious to engage in proofs which combine them in arguments as though the two were mergeable - that is, that one could begin with a finite set and recursively generate a denumerable set in the usual sense. This is a common flaw in proof schemes which use diagonalization as used by Kleene. The contradiction is obvious in such cases. One must have "completed the infinite" in order to derive some property which is then applied to the recursively enumerated members. And one good contradiction leads to another...

Paradox, that is.

SOURCES

NOTE: This paper consists of excerpts and clarifications of other papers prepared over the last decade. It should be clear from the titles how these paper contributed to the present effort.

KAPPA: A Logical Language, 1972

Quantum Logic and The Semantics of Natural Languages, 1975

Relativistic Quantum Logic and The Dynamics of Natural Language Acquisition, 1976

Methodologies in Multidisciplinary Modeling, 1979

A Lattice Theoretic Approach to Multidisciplinary Modeling: A Proposal, 1979

(with E. Oshins) Thoughts about Logic about Thoughts ... The Question: Schizophrenia?, 1979

Expressibility, Interpretation, and Paradox, 1980

On Certain Assumptions in the Proof of Godel's Second Incompleteness Theorem, 1981

Getting into Paradox... and Out Again: Discrete Steps Toward A Constructive Differential Topology, 1983

Some Comments on the Diagonal Method, 1984

The Halting Problem for the Combinatorial Hierarchy, 1984

A Decision Problem for Random Number Generators, 1984